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Duality Gaps in Stochastic Integer Programming

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Abstract. In this note, we explore the implications of a result that suggests that the duality gap caused by a Lagrangian relaxation of the nonanticipativity constraints in a stochastic mixed integer (binary) program diminishes as the number of scenarios increases. By way of an example, we illustrate that this is not the case. In general, the duality gap remains bounded away from zero.

1. Introduction

Stochastic integer programming problems arise in a variety of applications in which integer and combinatorial optimization problems are formulated in the presence of uncertainty. Examples of such applications arise in stochastic scheduling (Birge and Dempster, 1996), stochastic vehicle routing (Dror et al., 1989) and others (Dentcheva and Römisch, 1998); Carøe and Schultz, 1998). As one might expect, this class of problems inherits many of the difficulties associated with both stochastic linear programs and deterministic integer programs. As a result, for all but the smallest of instances, heuristic methods often provide the only practical approaches to these problems. One heuristic approach that is gaining in popularity combines a sample-based method with Lagrangian relaxation. The motivation for this approach lies in a result of Birge and Dempster (1996), where it is shown that for some problems, as the size of the sample space increases, the duality gap between a mixed integer 0-1 stochastic programs and a certain Lagrangian dual vanishes (see also Takriti, Long and Birge, 1996). One might be tempted to conjecture that this result might hold for stochastic integer programs allowing general integer variables. However, Carøe (1998) has provided a counterexample illustrating that the result cannot be extended to stochastic integer programs with general integer variables. Indeed, as shown below, the notion that the duality gap vanishes with an increasing sample space is erroneous even for 0-1 stochastic programs. Given the deep implications of Birge and Dempster (1996), our (negative) example is somewhat of a disappointment.

Table 1. Scenario constraints and their binary solutions

Scenario	Constraints	Binary solutions
s=1	$2_{x_{11}} + x_{21} \leqslant 2$ and $2_{x_{11}} - x_{21} \geqslant 0$	$(x_{11}, x_{21}) \in \{(0,0), (1,0)\}$
s=2	$x_{12} - x_{22} \geqslant 0.$	$(x_{12}, x_{22}) \in \{(0,0), (1,0), (1,1)\}$
s=3	$x_{13} + x_{23} \leqslant 1$	$(x_{13}, x_{23}) \in \{(0,0), (0,1), (1,0)\}$

2. Example

We begin our discussion with an example in which there are only three scenarios. We will verify that for this particular problem, the duality gap is strictly positive. We will then increase the size of the sample space in a manner that is consistent with the hypotheses of Birge and Dempster (1996), and verify that the gap remains bounded away from zero.

Consider a two-stage stochastic program with three equally likely scenarios, indexed by $s \in S = \{1, 2, 3\}$. In this example, one has to determine the value of two decision variables for each scenario: (x_{1s}, x_{2s}), $s \in S$. Within our example, the variables $\{x_{ls}\}_{s\in S}$ are 'first stage' variables, and thus are required to be constant across all scenarios (i.e., nonanticipative). The remaining variables, $\{x_{2s}\}_{s\in S}$ are 'second stage' variables, and thus are permitted to vary by scenario. Since the focus of our example is on binary decision variables, we will impose a binary restriction on all variables. Data associated with each of the scenarios is given in Table 1.

With all scenarios equally likely, and the objective of maximizing the expected value of the second stage variable, we have the following formulation.

$$\begin{array}{llllllll}
\text{Max} & & & & 1/3x_{21} & +1/3x_{22} & +1/3x_{23} & & \\
\text{s.t.} & 2x_{11} & & & +x_{21} & & & \leqslant 2 & (1a)\\
 & 2x_{11} & & & -x_{21} & & & \geqslant 0 & (1b)\\
 & & x_{12} & & & -x_{22} & & \geqslant 0 & (2)\\
 & & & x_{13} & & & +x_{23} & \leqslant 1 & (3)\\
 & \frac{2}{3}x_{11} & -\frac{1}{3}x_{12} & -\frac{1}{3}x_{13} & & & & =0 & (4)\\
 & & x_{is} \in \{0,1\} \quad i = 1,2 \quad \forall s \in S & & & & & &
\end{array}$$

The optimal value associated with this stochastic integer program is 1/3. Within the formulation, constraints (1)–(3) are those that are identified in Table 1. Constraint (4) is obtained from:

$$x_{11} - \{1/3x_{11} + 1/3x_{12} + 1/3x_{13}\} = 0,$$

and is known as a nonanticipativity constraint. As a result of the binary restrictions, it is clear that x_{11} can take the value 0 or 1 if, and only if, all of the first stage variables are equal to it. As we extend our example to include an increasing number of scenarios, the binary restrictions ensure that we may continue to represent the nonanticipativity requirements with a single constraint of this type.

Note that in the absence of the nonanticipativity restriction, (4), the stochastic integer program is separable by scenario. With this observation, the Birge and Dempster (1996) result revolves around the Lagrangian relaxation of this constraint. For notational convenience, let X_s denote the set of solutions specific to scenario s, including the binary restrictions. Thus, for example

$$\begin{aligned} X_3 &= \{(x_{13}, x_{23}) : x_{13} + x_{23} \leqslant 1, x_{i3} \in \{0,1\}, i = 1, 2\} \\ &= \{(0,0), (0,1), (1,0)\}. \end{aligned}$$

Then we may rewrite the problem as:

$$\begin{aligned} &\text{Max} \sum_{s=1}^{3} \frac{1}{3} x_{2s} \\ &\text{s.t.} (x_{1s}, x_{2s}) \in X_s \quad \forall s \in S \\ &\frac{2}{3} x_{11} - \sum_{s=2}^{3} \frac{1}{3} x_{1s} = 0 \end{aligned}$$

Relaxing the nonanticipativity constraint, let

$$\begin{aligned} D(\lambda) &= \text{Max}_{(x_{1s}, x_{2s}) \in X_s,\ \forall s \in S} \left\{ \frac{2\lambda}{3} x_{11} + \frac{1}{3} x_{21} - \frac{\lambda}{3} x_{12} + \frac{1}{3} x_{22} - \frac{\lambda}{3} x_{13} + \frac{1}{3} x_{23} \right\} \\ &= \text{Max}_{(x_{11}, x_{21}) \in X_1} \left\{ \frac{2}{3} \lambda x_{11} + \frac{1}{3} x_{21} \right\} + \text{Max}_{(x_{12}, x_{22})} \left\{ -\frac{1}{3} \lambda x_{12} + \frac{1}{3} x_{22} \right\} + \\ &\quad \text{Max}_{(x_{13}, x_{23}) \in X_3} \left\{ -\frac{1}{3} \lambda x_{13} + \frac{1}{3} x_{23} \right\} \end{aligned}$$

Using the solution sets provided in Table 1, we have

$$D(\lambda) = \text{Max}\,\{0, 2\lambda/3\} + \text{Max}\,\{0, (-\lambda + 1)/3\} + \text{Max}\,\{1/3, -\lambda/3\}$$

or equivalently:

$$D(\lambda) = \begin{cases} \frac{1}{3} - \frac{2}{3}\lambda & \lambda \le -1 \\ \frac{2}{3} - \frac{1}{3}\lambda & -1 \le \lambda \le 0 \\ \frac{2}{3} + \frac{1}{3}\lambda & 0 \le \lambda \le 1 \\ \frac{1}{3} + \frac{2}{3}\lambda & 1 \le \lambda \end{cases}$$

Graphically, we may depict this as shown in Figure 1.

The dual problem,

$$Min_{\lambda} D(\lambda)$$

has an optimal value of 2/3, which is achieved at λ=0. Given a primal value of 1/3, we note the presence of a duality gap.

Note that our example satisfies the conditions of the result in question; that is,

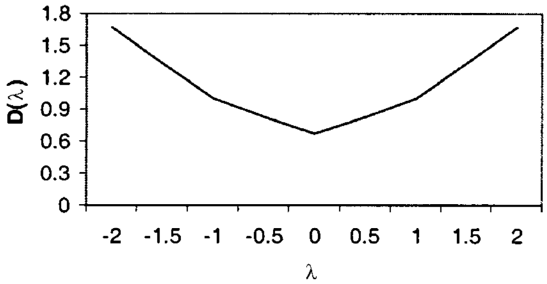

Figure 1. The dual objective function.

- X_s is compact for all $s \in S$
- The primal objective function is continuous
- X_s includes all scenario specific feasibility conditions (including the binary restrictions). As such, the extreme points of its convex hull are explicitly integer solutions

Under these conditions, the result being studied suggests that if all scenarios occur with equal probability, then as $\|S\| \to \infty$ the duality gap diminishes to zero. We will adapt our example to show that, in fact, this is *not* the case.

Suppose now that instead of 3 scenarios, we have $3k$ scenarios, each occurring with equal probability. Suppose further that the scenario specific constraints are as follows:

For $s \in \{1, \dots, k\}$,

$$X_s^k = \{(x_{1s}, x_{2s}) : (2-\epsilon_s)x_{1s} - x_{2s} \geqslant 0, (2-\epsilon_s)x_{1s} + x_{2s} \leqslant (2-\epsilon_s), x_{is} \in \{0,1\}, i = 1, 2\}$$

For $s \in \{k+1, \dots, 2k\}$,

$$X_s^k = \{(x_{1s}, x_{2s}) : x_{1s} - x_{2s} \geqslant -\epsilon_s, x_{is} \in \{0,1\}\}$$

For $s \in \{2k+1, \dots, 3k\}$,

$$X_s^k = \{(x_{1s}, x_{2s}) : x_{1s} + x_{2s} \leqslant 1 + \epsilon_s, x_{is} \in \{0,1\}\}$$

where $\epsilon_s = 1/2s$ for $s = 1, \dots, 3k$.

We may formulate this problem as:

$$\begin{aligned} &\text{Max} \sum_{s=1}^{3k} \frac{1}{3k} x_{2s} \\ &\text{s.t.} (x_{1s}, x_{2s}) \in X_s^k \quad s = 1, \dots, 3k \\ &\left(1 - \frac{1}{3k}\right) x_{11} - \sum_{k=2}^{3k} \frac{1}{3k} x_{1s} = 0. \end{aligned} \tag{5}$$

As before, the nonanticipativity constraint, (5), is derived from:

$$x_{11} - \sum_{s=1}^{3k} \frac{1}{3k} x_{1s} = 0.$$

Note that although the constraint coefficients have been slightly perturbed, the binary solutions remain the same as those previously identified. That is,

$$X_s^k = \begin{cases} X_1 & s = 1, \ldots, k \\ X_2 & s = k+1, \ldots, 2k \\ X_3 & s = 2k+1, \ldots, 3k. \end{cases}$$

As such, the primal objective value remains 1/3 for all values of $k \geq 1$. To investigate the duality gap, the dual problem is given by:

$$\begin{aligned} D_k(\lambda) =& \text{Max}_{(x_1,x_2)\in X_1} \left\{ \left(1 - \frac{1}{3k}\right) \lambda x_1 + \frac{1}{3k} x_2 \right\} + (k-1) \text{ Max }_{(x_1,x_2)\in X_1} \left\{ \frac{-\lambda}{3k} x_1 + \frac{1}{3k} x_2 \right\} \\ &+ k \text{ Max }_{(x_1,x_2)\in X_2} \left\{ \frac{-\lambda}{3k} x_1 + \frac{1}{3k} x_2 \right\} + k \text{ Max }_{(x_1,x_2)\in X_3} \left\{ \frac{-\lambda}{3k} x_1 + \frac{1}{3k} x_2 \right\} \\ =& \text{Max} \left\{ 0, \left(1 - \frac{1}{3k}\right) \lambda \right\} + (k-1) \text{Max} \left\{ 0, \frac{-\lambda}{3k} \right\} + k \text{ Max} \left\{ 0, \frac{1}{3k} + \frac{-\lambda}{3k} \right\} + k \text{ Max} \left\{ \frac{1}{3k}, \frac{-\lambda}{3k} \right\} \\ =& \text{Max} \left\{ 0, \left(1 - \frac{1}{3k}\right) \lambda \right\} + \text{Max} \left\{ 0, -\left(\frac{1}{3} - \frac{1}{3k}\right) \lambda \right\} + \text{Max} \left\{ 0, \frac{1}{3} - \frac{\lambda}{3} \right\} + \text{Max} \left\{ \frac{1}{3}, \frac{-\lambda}{3} \right\} \end{aligned}$$

That is,

$$D_k(\lambda) = \begin{cases} \frac{1}{3} - \left(1 - \frac{1}{3k}\right) \lambda & \lambda \leq -1 \\ \frac{2}{3} - \left(\frac{2}{3} - \frac{1}{3k}\right) \lambda & -1 \leq \lambda \leq 0 \\ \frac{2}{3} + \left(\frac{2}{3} - \frac{1}{3k}\right) \lambda & 0 \leq \lambda \leq 1 \\ \frac{1}{3} + \left(1 - \frac{1}{3k}\right) \lambda & 1 \leq \lambda \end{cases}$$

Independent of k, $D(\lambda)$ is minimized at $\lambda = 0$, with a dual objective value of 2/3. With a primal objective value of 1/3 for all values of k, we note the persistence of the duality gap of 1/3 as k increases, thereby contradicting the suggestion that the duality gap vanishes as k increases.

3. Conclusion

Given the counterexample, the question arises as to the nature of the error in Birge and Dempster (1996), who use the result due to Bertsekas (1982) (see Proposition 5.26) for the proof of their Theorem 6. When the linking constraints are of the form $\sum_s H_s x_s \leq b$, Bertsekas' result requires that for every s,

$$\forall x' \in \text{conv}(X_s) \exists x \in X \text{ such that } H_s x \leq H_s x'. \tag{6}$$

Theorem 6 of Birge and Dempster hypothesizes the same condition for linking constraints that are equalities. When the linking constraints are equalities, the analogous condition would replace the inequality in (6) with an equality. However, the implications of the nonanticipativity constraints of stochastic integer programs essentially require that X_s equals its convex hull. This cannot be satisfied unless the variables restricted to be integers in X_s have only one feasible value in X_s.

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